A Branch-and-Cut Approach for Simple Graph Partitioning on Sparse Graphs

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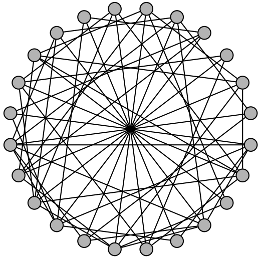
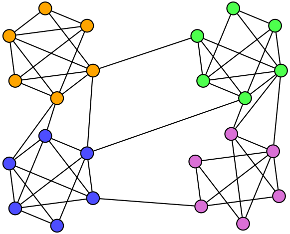
Problem Description

- The problem of graph partitioning refers to finding a partition of V such that the weight of the edges within each partition is maximized





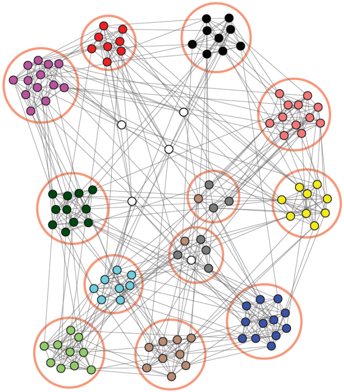
- Several versions of this problem arise by imposing constraints on the size, weight and number of partitions
- "Simple" graph partitioning refers to placing an upper bound on the weight of each partition

Motivation - Sparse Graphs

- Web graphs and social networks are sparse or locally dense

$$D = \frac{2|E|}{|V|(|V|-1)}$$

Name	Nodes	Edges	Density
LiveJournal	4.00E+06	3.47E+07	2.17E-06
Friendster	6.56E+07	1.81E+09	4.20E-07
Orkut	3.07E+06	1.17E+08	1.24E-05
Youtube	1.13E+06	2.99E+06	2.32E-06
DBLP	3.17E+05	1.05E+06	1.04E-05
Amazon	3.35E+05	9.26E+05	8.26E-06



[1] SNAP: A General-Purpose Network Analysis and Graph-Mining Library

Contents

Δ-Inequalities

$$x_{ij} = \begin{cases} 1 & \text{if } i, j \text{ in the same partition} \\ 0 & \text{otherwise} \end{cases}$$

c_e - weight of edge e
w_i - weight of vertex i
r - partition capacity

$$\max \quad \sum_{e \in E} c_e x_e$$

$$\begin{aligned} \text{s.t} \quad & x_{ij} + x_{jk} - x_{ik} \leq 1 && i, j, k \in \binom{V}{3} \\ & \sum_{j \in V \setminus \{i\}} w_j x_{ij} \leq r - w_i && i \in V \\ & x_{ij} \in \{0, 1\} && \{i, j\} \in E \\ & x_{ij} \in [0, 1] && \{i, j\} \notin E \end{aligned}$$

Node-to-Cluster Formulation

$$y_e = \begin{cases} 1 & \text{if } e \text{ is in the edge cut } \delta(V_1, \dots, V_K) \\ 0 & \text{otherwise} \end{cases} \quad , \quad z_i^k = \begin{cases} 1 & \text{if } i \text{ is in cluster } k \\ 0 & \text{otherwise} \end{cases}$$

w_i – the weight of vertex i

$$\begin{aligned} \min \quad & \sum_{e \in E} c_e y_e & \\ \text{s.t} \quad & \sum_{k=1}^{K} z_i^k = 1 & i \in V \\ & \sum_{i \in V} w_i z_i^k \leq r & k = 1, \dots, K \\ & y_{ij} \geq \sum_{k \in K_1} z_i^k - \sum_{k \in K_1} z_j^k & K_1 \subseteq \{1, \dots, K\}, \quad (i,j) \in E \\ & y_{ij} \in \{0, 1\} & \{i, j\} \in E \\ & z_i^k \in \{0, 1\} & i \in V, k = 1, \dots, K \end{aligned}$$

Issues with Node-Based Formulations

- Consider the constraint set from the Δ-Inequalities

$$x_{ij} + x_{jk} - x_{ik} \leq 1 \quad i, j, k \in \binom{V}{3}$$

- Though a compact $O(n^3)$ formulation, the classic partitioning formulation assumes a complete graph
- The node-to-cluster formulation is highly symmetric and as a result performs poorly during branch and bound
- We seek a formulation which is capable of exploiting the structure of sparse graphs
- We introduce several extended formulations and study their projection

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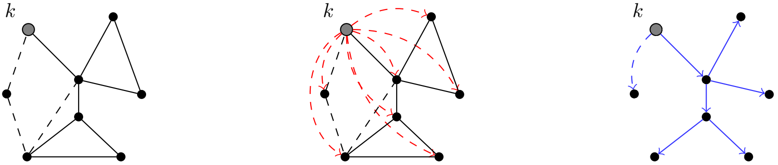
Max Flow - Formulation

- Given cut edge solution $\bar{\mathbf{y}}$, we wish to maximize the flow on the arcs leaving node k

$$
\begin{aligned}
MF(k,\bar{\mathbf{y}}) = \max \quad & \sum_{(k,i)\in A} f_{ki}^k + \sum_{(k,i)\in A'} 0 \cdot f_{ki}^k - \sum_{(i,j)\in A} f_{ij}^k \bar{y}_e \\
\text{s.t.} \quad & \sum_{(k,i)\in A} f_{ki}^k + \sum_{(k,i)\in A'} f_{ki}^k = \sum_{i\in V\setminus\{k\}} w_i \\
& f_{ki}^k - \sum_{j:(i,j)\in A} f_{ij}^k + \sum_{j:(j,i)\in A} f_{ji}^k = w_i && i \in V \setminus \{k\} \\
& f_{ij}^k \geq 0 && (i,j) \in A \cup A'
\end{aligned}
$$



Max Flow

- Let $\text{MF}(k, \bar{\mathbf{y}})$ represent the optimal solution for the max flow problem at node k
- For $k \in V$, if $\text{MF}(k, \bar{\mathbf{y}}) \leq r - w_k$, then the $\bar{\mathbf{y}}$ is a feasible edge cut
- We can utilize $\text{MF}(k, \mathbf{y})$ as inner problem, and take its dual in order to obtain the single level optimization model

$$
\begin{aligned}
\min \quad & \sum_{e \in E} c_e y_e & \\
\text{s.t.} \quad & \text{MF}(k, \mathbf{y}) \leq r - w_k & k \in V \\
& y_e \in \{0, 1\} & e \in E
\end{aligned}
$$

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$$
\begin{aligned}
\min \quad & \sum_{e \in E} c_e y_e & \\
\text{s.t.} \quad & \sum_{i \in V \setminus \{k\}} w_i \alpha_i^k \leq r - w_k & k \in V \\
& \alpha_i^k + y_e \geq 1 & k \in V,\ i \in N(k) \\
& \alpha_i^k - \alpha_j^k + y_e \geq 0 & e = \{i, j\} \in E, k \in V \setminus \{i, j\} \\
& \alpha_j^k - \alpha_i^k + y_e \geq 0 & e = \{i, j\} \in E, k \in V \setminus \{i, j\} \\
& \alpha_i^k \geq 0 & k \in V,\ i \in V \setminus \{k\} \\
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& \alpha_i^k + y_e \geq 1 & k \in V,\ i \in N(k) \\
\rightarrow\ & \alpha_i^k - \alpha_j^k + y_e \geq 0 & e = \{i,j\} \in E, k \in V \setminus \{i,j\} \\
\rightarrow\ & \alpha_j^k - \alpha_i^k + y_e \geq 0 & e = \{i,j\} \in E, k \in V \setminus \{i,j\} \\
& \alpha_i^k \geq 0 & k \in V,\ i \in V \setminus \{k\} \\
& y_e \in \{0,1\} & e \in E
\end{aligned}
$$

Path Formulation

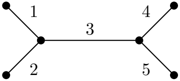
$$x_{ij} = \begin{cases} 1 & \text{if vertices } i,j \text{ remain connected in the edge cut} \\ 0 & \text{otherwise} \end{cases}$$

$$\begin{aligned}
\min \quad & \sum_{e \in E} c_e y_e \\
\text{s.t.} \quad & x_{ij} + \sum_{e \in P^{ij}} y_e \geq 1 && i,j \in \binom{V}{2}, P^{ij} \in \mathcal{P} \\
& \sum_{j \in V \setminus \{i\}} w_j x_{ij} \leq r - w_i && i \in V \\
& y_{ij} \in [0,1] && i,j \in \binom{V}{2} \\
& y_e \in \{0,1\} && e \in E
\end{aligned}$$

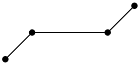
- The size of the path inequalities is exponential but for sparse graphs only a fraction are necessary in the optimal solution
- We can perform lazy and fractional separation of these inequalities via Dijkstra's with min heap

Studying the Projection of PATH and MF

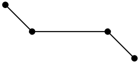
- The projection of (PATH) into the space of the edges produces similar extreme points to that of (MF)
- Consider the following graph with $w_i = 1$ for $i \in V$ and $r = 3$.



$y_1 + y_3 + y_4 \geq 1$

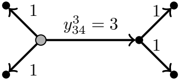


$y_2 + y_3 + y_4 \geq 1$



$y_1 + y_3 + y_5 \geq 1$

$y_2 + y_3 + y_5 \geq 1$



$y_1 + y_2 + 3y_3 + y_4 + y_5 \geq 3$

- It turns out that the strength of the relaxation of (PATH) is just as strong as that of (MF) and as a result of equivalent strength to (Δ)

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SGPP on Trees

Definition

A tree T is a minimal tree cover if $w(T) > r$, but $w(T - \{e\}) \leq r$ for $e \in E(T)$. Let Ω_T represent the set of minimal tree covers for an instance SGPP(r)

Theorem

An edge-cut $S \subseteq E$ is a feasible solution for SGPP(r) if and only if for all $T \in \Omega_T$, $S \cap E(T) \neq \emptyset$

- Therefore, the following set cover formulation is valid for SGPP

$$
\begin{aligned}
& \min && \sum_{e \in E} c_e y_e \\
\text{(SGPP-T)}\quad & \text{s.t.} && \sum_{e \in E(T)} y_e \geq 1 \quad T \in \Omega_T \\
& && y_e \in \{0,1\} \quad e \in E
\end{aligned}
$$

Properties of (SGPP-T)

Theorem

Given a simple, nonempty, connected graph $G = (V, E)$, a weight vector $\mathbf{w} \in \mathbb{Z}_+^n$, edge cut cost $\mathbf{c} \in \mathbb{R}_+^m$ and a scalar integer r such that $w_i + w_j \leq r$ for all $\{i, j\} \in E$, the following statements are true:

1. *$\mathcal{P}$ is full-dimensional*
2. *The trivial inequality $y_e \leq 1$ is a facet*
3. *Let e be represented by the node pair $\{i, j\}$. Then $y_e \geq 0$ is a facet if and only if for all $k \in N(i) \cup N(j) \setminus \{i, j\}$, $w_i + w_j + w_k \leq r$.*
4. *Given minimal cover tree $T \in \Omega_T(r)$, the corresponding minimal cover tree inequality $\sum_{e \in E(T)} y_e \geq 1$ induces a facet of $(\mathcal{P})$ if and only if:*
   - 4.1 *there exists no edge $e \in E$ such that if added to T, completes a cycle in G with $|V(T)|$ nodes*
   - 4.2 *there exists no edge $e \in E$ containing node $k \notin V(T)$ that is incident to an internal node $v \in V(T)$ such that the tree rooted at v does not have a branch B such that $w(T) + w_k - w(B) \leq r$*

Separating the Minimum Cover Tree Inequalities

- The projections of (Δ), (MF) and (PATH) are incapable of producing the minimal tree cover inequalities
- The size of Ω_T grows prohibitively large as G increases in size and density

Theorem

Given a cut-edge solution $\mathbf{y}$, separating the minimal cover tree inequalities is:

1. *NP-Hard if $\bar{\mathbf{y}} \in [0,1]^m$*
2. *solvable in $O(n+m)$ if $\bar{\mathbf{y}} \in \{0,1\}^m$.*

- We introduce a dynamic programming algorithm whose linear programming extension provides a separation algorithm for generating tree inequalities

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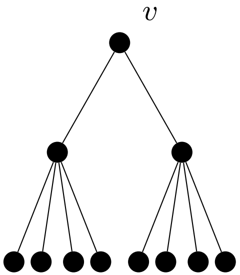
Dynamic Program on Trees

- We develop an $O(nr^2)$ dynamic programming algorithm to solve the SGPP on trees
- We are not the first to develop such an algorithm for partitioning trees (Lukes, J.A. 1974)
- However, our DP is extensible to a linear program while still maintaining the same complexity
- Our goal is to utilize the extreme rays of this linear program as a separation procedure for generating tree inequalities
- For the purpose of simplicity, we will describe the cardinality version of the DP and show its weighted version after

Notation and Description

- Let T be a tree rooted at vertex v
- We say that T_i represents the subtree rooted at vertex i
- We say the closed neighborhood of i, contains itself and all of the vertices in T_i which are connected to i
- We denote the closed neighborhood of i as o_i



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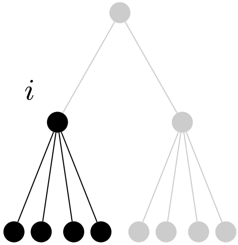
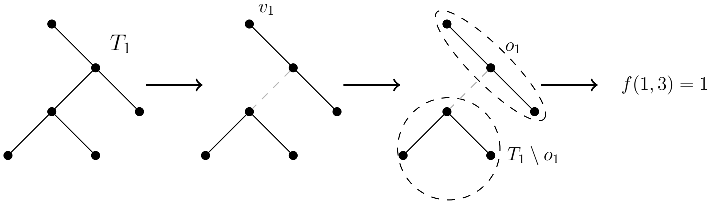


Notation and Description

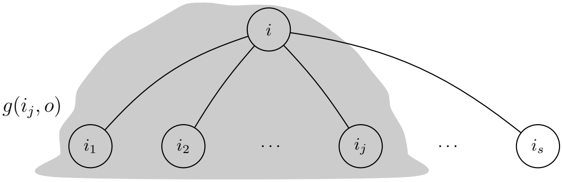
- Let $f(i, o)$ represent the minimum number of edges to be removed such that the closed neighborhood of i contains o nodes, and the largest component in $T_i \setminus o_i$ has size at most r
- Suppose $r = 3$



- Base Case:
  - $f(i, 1) = 0$ if i is a leaf
  - $f(i, o) = \infty$ if $o > |T_i|$
  - $f(i, 0) = \infty$

Notation and Description

- Suppose vertex i has s children
- Let $g(i_j, o)$ represent the minimum number of edges cut from $T_{i_1}, \ldots, T_{i_j}$ and $(i, i_1), \ldots, (i, i_j)$ such that the closed neighborhood of i is o, and the largest component in $T_{i_1} \setminus o_{i_1}, \ldots, T_{i_j} \setminus o_{i_j}$ has size at most r



- When $j = s$ it follows that:

$$f(i, o) = g(i_s, o)$$

Dynamic Program on Trees

$$f(i, o) = g(i_s, o)$$

$$g(i_1, 1) = 1 + \min_{k=1}^{r}\{f(i_1, k)\}$$

$$g(i_j, 1) = 1 + g(i_{j-1}, 1) + \min_{k=1}^{r}\{f(i_j, k)\}$$

$$g(i_1, o) = f(i, o-1)$$

$$g(i_j, o) = \min \begin{cases} 1 + g(i_{j-1}, o) + \min_{k=1}^{r}\{f(i_j, k)\} \\ \min_{k=1}^{o}\{g(i_{j-1}, k) + f(i_j, o-k)\} \end{cases}$$

- Space complexity: $f = O(nr)$, $g = O(nr)$
- Time complexity: $O(nr^2)$

Linear Programming Formulation

I – the set of interior nodes (i.e. V without leaves)

$S(i)$ – the children of node i

f_{io} – a representation of $f(i, o)$

$g_{i_j o}$ – a representation of $g(i_j, o)$

$$
\begin{array}{lll}
\max & z & \\
\text{s.t.} & z \leq f_{vo} & v \text{ - root}, o = 1, \dots, r \\
& f_{io} = g_{i_s o} & i \in I, o = 1, \dots, r \\
& g_{i_j o} \leq 1 + g_{i_{j-1} o} + f_{i_j k} & i \in I, j \in S(i), o = 1, \dots, r, k = 1, \dots, r \\
& g_{i_j o} \leq g_{i_{j-1} k} + f_{i_j o-k} & i \in I, j \in S(i), o = 1, \dots, r, k = 1, \dots, o \\
& f_{io} \geq 0, \quad g_{i_j o} \geq 0 &
\end{array}
$$

Linear Programming Separation

- We can separate a fractional solution SGPP formulation with the modified LP

$$
\begin{aligned}
\max \quad & z - \sum_{e \in E} \bar{y}_e h_e \\
\text{s.t.} \quad & z \le f_{vo} && v \text{ - root}, o = 1, \ldots, r \\
& f_{io} = g_{i_s o} && i \in I, o = 1, \ldots, r \\
& g_{i_j o} \le 1 + g_{i_{j-1} o} + f_{i_j k} + h_e && i \in I, j \in S(i), o = 1, \ldots, r, k = 1, \ldots, r \\
& g_{i_j o} \le g_{i_{j-1} k} + f_{i_j o-k} && i \in I, j \in S(i), o = 1, \ldots, r, k = 1, \ldots, o \\
& f_{io} \ge 0, \quad g_{i_j o} \ge 0, \quad h_e \ge 0
\end{aligned}
$$

- For a given $\bar{\mathbf{y}}$, if this formulation is unbounded, then there is a violating tree in our current solution

- The extreme direction produced by this unbounded solution is used to generate a cut in the form

$$\sum_{e \in E} \bar{h}_e y_e \ge \bar{z}$$

Contents

Computational Study

- The computational study was performed on a computing cluster with 12 nodes, each featuring a 12-core Intel Xeon® E5-2620 v3 2.4 GHz processor, 128 GB of RAM, and running `Linux x86_64, CentOS 7.2`
- All formulations and algorithms were implemented in `C++`, compiled with `GCC 7.3.0` and solved using `Gurobi 9.0.0`
- Instances are selected from real social networks and randomly generated networks (ERG, BA, WS)
- Weighted instances were generated by selecting $w_i \sim U(5, 15)$ for $i \in V$
- Parameter r was selected as 10, 20, and 30 percent of $|V|$ or $\sum_{i \in V} w_i$

WS-Unweighted, 30 Nodes

			Time				Gap		
n	m	r	Triangle	Flow	Path	obj	Triangle	Flow	Path
30	60	3	0.27	**0.18**	0.21	32	0.00	0.00	0.00
30	60	3	0.41	0.17	**0.12**	34	0.00	0.00	0.00
30	60	3	0.25	0.19	**0.16**	33	0.00	0.00	0.00
30	60	6	0.52	0.34	0.34	18	0.00	0.00	0.00
30	60	6	1.02	0.47	**0.35**	21	0.00	0.00	0.00
30	60	6	1.04	0.62	**0.39**	20	0.00	0.00	0.00
30	60	9	1.74	0.34	**0.19**	14	0.00	0.00	0.00
30	60	9	1.47	**0.19**	0.77	16	0.00	0.00	0.00
30	60	9	3.79	1.59	**0.93**	16	0.00	0.00	0.00
30	90	3	0.34	**0.23**	1.03	61	0.00	0.00	0.00
30	90	3	0.96	0.83	**0.79**	62	0.00	0.00	0.00
30	90	3	0.46	**0.34**	0.52	61	0.00	0.00	0.00
30	90	6	1.43	**0.85**	1.04	38	0.00	0.00	0.00
30	90	6	3.31	**1.80**	4.47	41	0.00	0.00	0.00
30	90	6	2.71	1.59	**0.82**	38	0.00	0.00	0.00
30	90	9	5.80	**3.64**	5.78	31	0.00	0.00	0.00
30	90	9	3.63	**1.89**	3.00	32	0.00	0.00	0.00
30	90	9	3.64	2.36	**1.32**	29	0.00	0.00	0.00
30	180	3	**0.17**	0.19	0.79	150	0.00	0.00	0.00
30	180	3	**0.16**	0.19	0.62	150	0.00	0.00	0.00
30	180	3	**0.17**	0.19	1.09	150	0.00	0.00	0.00
30	180	6	**3.19**	3.93	36.91	111	0.00	0.00	0.00
30	180	6	5.45	**4.07**	45.46	113	0.00	0.00	0.00
30	180	6	**3.07**	3.96	47.82	110	0.00	0.00	0.00
30	180	9	**19.09**	20.16	136.97	89	0.00	0.00	0.00
30	180	9	**16.03**	17.95	112.50	90	0.00	0.00	0.00
30	180	9	**11.21**	17.61	50.75	87	0.00	0.00	0.00

WS-Weighted, 30 Nodes

			Time				Gap		
n	m	r	Triangle	Flow	Path	obj	Triangle	Flow	Path
30	60	25	0.87	0.41	**0.15**	36	0.00	0.00	0.00
30	60	25	0.74	0.42	**0.22**	38	0.00	0.00	0.00
30	60	25	1.00	0.41	**0.33**	38	0.00	0.00	0.00
30	60	50	3.25	1.24	**0.54**	20	0.00	0.00	0.00
30	60	50	5.06	2.07	**1.49**	23	0.00	0.00	0.00
30	60	50	2.16	0.56	**0.51**	21	0.00	0.00	0.00
30	60	75	0.75	0.32	**0.22**	14	0.00	0.00	0.00
30	60	75	3.59	1.79	**0.72**	17	0.00	0.00	0.00
30	60	75	3.41	1.84	**1.33**	16	0.00	0.00	0.00
30	90	25	0.93	0.93	**0.17**	63	0.00	0.00	0.00
30	90	25	2.15	1.33	**0.86**	64	0.00	0.00	0.00
30	90	25	1.46	1.00	**0.51**	64	0.00	0.00	0.00
30	90	50	6.51	**4.88**	20.51	41	0.00	0.00	0.00
30	90	50	17.46	**13.28**	23.50	44	0.00	0.00	0.00
30	90	50	6.38	**4.40**	8.18	41	0.00	0.00	0.00
30	90	75	7.65	**3.59**	4.58	31	0.00	0.00	0.00
30	90	75	4.22	2.61	**0.77**	32	0.00	0.00	0.00
30	90	75	7.98	**4.24**	5.03	30	0.00	0.00	0.00
30	180	25	17.69	**13.99**	34.79	151	0.00	0.00	0.00
30	180	25	**13.82**	14.24	18.48	151	0.00	0.00	0.00
30	180	25	**7.35**	8.52	10.64	151	0.00	0.00	0.00
30	180	50	43.31	**34.71**	379.84	114	0.00	0.00	0.00
30	180	50	315.60	**84.39**	4324.67	116	0.00	0.00	0.00
30	180	50	89.83	**63.73**	2783.36	115	0.00	0.00	0.00
30	180	75	**27.18**	29.05	228.73	89	0.00	0.00	0.00
30	180	75	**62.95**	66.15	5676.66	92	0.00	0.00	0.00
30	180	75	27.53	**24.72**	247.32	89	0.00	0.00	0.00

WS-Unweighted, 60 Nodes

			Time				Gap		
n	m	r	Triangle	Flow	Path	obj	Triangle	Flow	Path
60	120	6	52.03	**8.86**	53.04	41	0.00	0.00	0.00
60	120	6	477.03	**11.68**	43.62	45	0.00	0.00	0.00
60	120	6	79.48	11.53	**10.64**	40	0.00	0.00	0.00
60	120	12	7200.04	44.17	**33.22**	26	0.01	0.00	0.00
60	120	12	2709.17	**23.51**	33.65	29	0.00	0.00	0.00
60	120	12	135.77	**1.51**	4.83	23	0.00	0.00	0.00
60	120	18	7200.01	**10.86**	17.85	18	0.01	0.00	0.00
60	120	18	7200.04	**65.94**	198.35	24	0.04	0.00	0.00
60	120	18	7200.02	18.04	**14.56**	19	0.01	0.00	0.00
60	180	6	20.38	**6.14**	27.93	77	0.00	0.00	0.00
60	180	6	171.11	**35.63**	117.55	82	0.00	0.00	0.00
60	180	6	26.61	**9.17**	55.59	78	0.00	0.00	0.00
60	180	12	1011.38	**50.40**	119.55	49	0.00	0.00	0.00
60	180	12	1261.06	261.55	**59.53**	53	0.00	0.00	0.00
60	180	12	2504.33	344.06	**205.16**	51	0.00	0.00	0.00
60	180	18	7200.01	**305.80**	416.37	38	0.02	0.00	0.00
60	180	18	7200.03	**414.78**	878.62	44	0.02	0.00	0.00
60	180	18	3784.12	**229.35**	433.17	39	0.00	0.00	0.00
60	360	6	1567.61	**175.17**	7207.22	228	0.00	0.00	0.04
60	360	6	1698.36	**197.19**	7206.29	229	0.00	0.00	0.01
60	360	6	591.47	**124.63**	7204.17	225	0.00	0.00	0.03
60	360	12	198.82	130.12	**30.93**	144	0.00	0.00	0.00
60	360	12	700.59	**165.82**	483.61	149	0.00	0.00	0.00
60	360	12	164.56	**18.44**	163.56	142	0.00	0.00	0.00
60	360	18	7200.03	**3811.82**	7206.36	117	0.03	0.00	0.04
60	360	18	7200.02	**4124.89**	7202.84	120	0.03	0.00	0.04
60	360	18	7200.01	**5124.41**	7207.26	116	0.03	0.00	0.07

WS-Weighted, 60 Nodes

			Time				Gap		
n	m	r	Triangle	Flow	Path	Obj	Triangle	Flow	Path
60	120	54	128.66	26.82	**25.57**	41	0.00	0.00	0.00
60	120	54	2094.27	**181.15**	743.11	47	0.00	0.00	0.00
60	120	54	542.26	**42.60**	86.73	43	0.00	0.00	0.00
60	120	108	376.11	**25.65**	68.73	25	0.00	0.00	0.00
60	120	108	7200.01	**138.27**	149.35	31	0.02	0.00	0.00
60	120	108	3867.57	**37.12**	72.53	26	0.00	0.00	0.00
60	120	162	735.12	15.72	**7.62**	17	0.00	0.00	0.00
60	120	162	3438.90	**24.11**	96.58	23	0.00	0.00	0.00
60	120	162	4837.30	25.68	24.92	19	0.00	0.00	0.00
60	180	54	3061.14	**700.48**	1234.59	81	0.00	0.00	0.00
60	180	54	7200.02	**1950.51**	2902.86	85	0.01	0.00	0.00
60	180	54	591.84	**166.72**	795.84	82	0.00	0.00	0.00
60	180	108	7200.01	1182.03	**791.13**	51	0.02	0.00	0.00
60	180	108	7200.01	920.85	**494.54**	56	0.01	0.00	0.00
60	180	108	7200.02	902.57	**844.57**	53	0.02	0.00	0.00
60	180	162	4728.82	211.50	**120.69**	37	0.00	0.00	0.00
60	180	162	5553.63	**237.51**	815.36	43	0.00	0.00	0.00
60	180	162	5995.95	270.65	**219.90**	40	0.00	0.00	0.00
60	360	54	7200.03	7200.05	7208.53	-	0.05	**0.02**	0.03
60	360	54	7200.05	7200.05	7220.43	-	0.06	**0.02**	0.04
60	360	54	7200.05	7200.05	7225.59	-	0.08	**0.05**	0.05
60	360	108	2936.37	**1129.04**	3387.42	148	0.00	0.00	0.00
60	360	108	7200.05	**4004.85**	7210.35	153	0.01	0.00	0.03
60	360	162	7200.03	**2744.07**	7203.16	117	0.03	0.00	0.03
60	360	162	7200.02	**6972.54**	7202.78	120	0.03	0.00	0.06

BA-Unweighted, 30 Nodes

			Time				Gap		
n	m	r	Triangle	Flow	Path	Optimal	Triangle	Flow	Path
30	56	3	0.15	0.10	**0.07**	37	0.00	0.00	0.00
30	56	3	0.37	0.16	0.17	36	0.00	0.00	0.00
30	56	3	0.41	0.14	**0.05**	36	0.00	0.00	0.00
30	56	6	3.50	0.37	0.34	26	0.00	0.00	0.00
30	56	6	7.08	1.19	**0.44**	26	0.00	0.00	0.00
30	56	6	3.39	0.60	**0.35**	26	0.00	0.00	0.00
30	56	9	6.45	**1.25**	1.69	21	0.00	0.00	0.00
30	56	9	2.92	**0.36**	0.69	20	0.00	0.00	0.00
30	56	9	6.74	1.64	**0.86**	22	0.00	0.00	0.00
30	81	3	0.49	0.27	**0.19**	57	0.00	0.00	0.00
30	81	3	1.39	**0.37**	0.43	57	0.00	0.00	0.00
30	81	3	1.02	0.30	**0.24**	56	0.00	0.00	0.00
30	81	6	7.81	1.68	**1.60**	44	0.00	0.00	0.00
30	81	6	5.49	**1.09**	2.21	43	0.00	0.00	0.00
30	81	6	12.25	**1.92**	6.18	44	0.00	0.00	0.00
30	81	9	6.31	2.20	2.10	35	0.00	0.00	0.00
30	81	9	10.75	4.29	**3.72**	35	0.00	0.00	0.00
30	81	9	14.72	**2.89**	12.54	36	0.00	0.00	0.00
30	176	3	0.31	**0.18**	0.43	146	0.00	0.00	0.00
30	176	3	0.20	**0.46**	0.60	146	0.00	0.00	0.00
30	176	3	0.19	**0.45**	1.00	146	0.00	0.00	0.00
30	176	6	5.34	**3.01**	55.13	118	0.00	0.00	0.00
30	176	6	11.91	**7.38**	89.08	116	0.00	0.00	0.00
30	176	6	27.58	**6.77**	78.67	118	0.00	0.00	0.00
30	176	9	17.33	**8.27**	44.04	99	0.00	0.00	0.00
30	176	9	303.59	**84.61**	4230.98	100	0.00	0.00	0.00
30	176	9	291.08	**61.64**	821.22	101	0.00	0.00	0.00

BA-Weighted, 30 Nodes

			Time				Gap		
n	m	r	Triangle	Flow	Path	Obj	Triangle	Flow	Path
30	56	25	0.66	0.21	**0.15**	38	0.00	0.00	0.00
30	56	25	0.74	0.32	**0.16**	37	0.00	0.00	0.00
30	56	25	1.11	0.47	**0.22**	39	0.00	0.00	0.00
30	56	50	8.93	0.30	**0.35**	26	0.00	0.00	0.00
30	56	50	18.35	2.14	**1.96**	27	0.00	0.00	0.00
30	56	50	4.56	0.77	**0.44**	27	0.00	0.00	0.00
30	56	75	7.80	1.46	**0.89**	21	0.00	0.00	0.00
30	56	75	6.76	1.35	**0.92**	21	0.00	0.00	0.00
30	56	75	7.97	1.65	**0.79**	22	0.00	0.00	0.00
30	81	25	2.04	0.56	**0.36**	59	0.00	0.00	0.00
30	81	25	2.70	1.44	**0.34**	58	0.00	0.00	0.00
30	81	25	3.46	1.37	**0.62**	60	0.00	0.00	0.00
30	81	50	21.93	**3.17**	5.38	45	0.00	0.00	0.00
30	81	50	23.20	**3.73**	7.08	44	0.00	0.00	0.00
30	81	50	16.84	**2.16**	5.92	44	0.00	0.00	0.00
30	81	75	12.54	2.27	**2.01**	35	0.00	0.00	0.00
30	81	75	34.25	**7.11**	33.75	36	0.00	0.00	0.00
30	81	75	20.54	**3.50**	22.37	36	0.00	0.00	0.00
30	176	25	2.88	3.15	**2.07**	148	0.00	0.00	0.00
30	176	25	**9.45**	10.20	11.37	148	0.00	0.00	0.00
30	176	25	3.53	4.25	3.65	148	0.00	0.00	0.00
30	176	50	41.52	**18.38**	254.36	119	0.00	0.00	0.00
30	176	50	282.20	**97.42**	2558.82	119	0.00	0.00	0.00
30	176	50	862.98	**336.87**	7213.74	121	0.00	0.00	0.02
30	176	75	51.41	**26.98**	368.15	100	0.00	0.00	0.00
30	176	75	188.47	**104.02**	3828.51	100	0.00	0.00	0.00
30	176	75	822.35	**167.31**	7216.76	102	0.00	0.00	0.02

BA-Unweighted, 60 Nodes

			Time				Gap		
n	m	r	Triangle	Flow	Path	Obj	Triangle	Flow	Path
60	116	6	2094.86	**7.90**	26.64	59	0.00	0.00	0.00
60	116	6	3327.02	**10.77**	79.21	60	0.00	0.00	0.00
60	116	6	888.88	**3.90**	13.34	60	0.00	0.00	0.00
60	116	12	7200.01	**25.17**	127.35	44	0.03	0.00	0.00
60	116	12	7200.01	**26.93**	200.21	44	0.04	0.00	0.00
60	116	12	7200.01	**20.61**	81.05	45	0.01	0.00	0.00
60	116	18	7200.01	**62.11**	396.65	36	0.04	0.00	0.00
60	116	18	7200.03	**56.43**	350.51	36	0.03	0.00	0.00
60	116	18	7200.01	**58.13**	229.43	38	0.05	0.00	0.00
60	171	6	4570.05	**15.23**	1611.19	101	0.00	0.00	0.00
60	171	6	6693.56	**24.12**	5973.93	99	0.00	0.00	0.00
60	171	6	7200.04	**51.77**	7200.68	100	0.03	0.00	0.02
60	171	12	7200.03	**321.15**	2917.14	77	0.02	0.00	0.00
60	171	12	7200.02	**559.21**	7202.85	78	0.04	0.00	0.08
60	171	12	7200.02	**164.27**	5120.93	76	0.02	0.00	0.00
60	171	18	7200.03	**1116.33**	7200.08	65	0.06	0.00	0.03
60	171	18	7200.03	**611.45**	2569.33	62	0.03	0.00	0.00
60	171	18	7200.04	**3316.31**	7200.90	65	0.06	0.00	0.05
60	371	6	7200.03	7200.04	7206.76	-	0.16	**0.04**	0.05
60	371	6	7200.08	7200.03	7203.87	-	0.13	**0.03**	0.03
60	371	6	7200.06	7200.02	7207.71	-	0.11	**0.03**	0.04
60	371	12	7200.01	7200.01	7203.60	-	0.15	**0.07**	0.16
60	371	12	7200.05	7200.04	7201.93	-	0.17	**0.05**	0.18
60	371	12	7202.76	7200.03	7203.04	-	0.14	**0.05**	0.13
60	371	18	7200.06	7200.06	7201.08	-	0.17	**0.07**	0.18
60	371	18	7200.06	7200.01	7200.60	-	0.15	**0.08**	0.29
60	371	18	7200.02	7200.01	7200.86	-	0.12	**0.07**	0.21

BA-Weighted, 60 Nodes

			Time				Gap		
n	m	r	Triangle	Flow	Path	Obj	Triangle	Flow	Path
60	116	54	3887.85	**15.22**	44.97	59	0.00	0.00	0.00
60	116	54	7200.07	198.33	**128.32**	60	0.04	0.00	0.00
60	116	54	2943.12	**17.43**	29.73	60	0.00	0.00	0.00
60	116	108	7200.03	**53.46**	148.06	44	0.04	0.00	0.00
60	116	108	7200.03	**62.47**	391.35	45	0.04	0.00	0.00
60	116	108	7200.03	**66.24**	163.66	46	0.04	0.00	0.00
60	116	162	7200.01	**122.66**	466.90	36	0.04	0.00	0.00
60	116	162	7200.03	**40.94**	326.74	36	0.03	0.00	0.00
60	116	162	7200.01	**76.71**	278.03	38	0.04	0.00	0.00
60	171	54	7200.10	**4092.24**	7202.80	104	0.06	0.00	0.04
60	171	54	7200.02	**1231.54**	7200.17	100	0.04	0.00	0.04
60	171	54	7200.04	**3098.09**	7203.11	102	0.06	0.00	0.05
60	171	108	7200.03	**595.54**	7202.15	79	0.03	0.00	0.05
60	171	108	7200.02	**1328.73**	7203.44	79	0.07	0.00	0.08
60	171	108	7200.01	**5899.18**	7205.47	78	0.05	0.00	0.06
60	171	162	7200.01	**1312.88**	7201.26	66	0.07	0.00	0.06
60	171	162	7200.01	**1464.85**	5133.43	63	0.04	0.00	0.00
60	171	162	7200.01	**1761.78**	7206.11	65	0.07	0.00	0.06
60	371	54	7200.03	7200.05	7219.02	277*	0.18	0.09	0.06

Closing Remarks and Future Work

- Conclusion
  - We develop several new mathematical programming approaches for solving SGPP on sparse graphs
  - We characterize the problem over the trees of the G
  - In general, the path formulation performs better on sparse instances while the flow formulation performs better on denser instances
- Future work
  - Describe necessary and sufficient conditions for when inequalities on larger trees form facets
  - Develop a heuristic separation for the minimal cover tree inequalities and for larger trees

Q&A

Thank you for attending!

GAMMA

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